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Rainbow Dynamic Coloring in Fan and Star-fan Graphs

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Abstract. Let G be a simple, non-trivial, connected graph that admits a vertex coloring $c : V(G) \to \{1, 2, 3, \ldots, k\}$, where $k \in \mathbb{N}$. A rainbow dynamic coloring is a dynamic coloring in which every pair of vertices is connected by at least one path whose internal vertices receive distinct colors. The rainbow dynamic coloring number of G, denoted by $rdyc(G)$, is the minimum number k for which such a coloring exists.
In this paper, we determine the rainbow dynamic coloring number for certain classes of graphs, namely fan graphs and star-fan graphs. In particular, we obtain exact values and bounds for $rdyc(G)$ for these graph families. These results highlight the influence of graph structure on rainbow dynamic coloring and contribute to the study of rainbow dynamic coloring parameters.

Key Words and Phrases: Dynamic coloring, rainbow vertex coloring, rainbow dynamic coloring.

2010 Mathematics Subject Classifications: 05C15, 05C69, 05C76.

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1. Introduction

In graph theory, vertex coloring plays an important role in understanding the structural properties of graphs and in solving many practical problems such as scheduling, frequency assignment, and network design. Over time, several variations of vertex coloring have been introduced to impose additional constraints and model more complex situations. These include dynamic coloring and rainbow vertex coloring, both of which extend the classical concept of proper coloring. Dynamic coloring aims to ensure color diversity in the neighborhood of each vertex, while rainbow vertex coloring focuses on the existence of paths whose internal vertices receive distinct colors [1, 2].

Dynamic coloring was introduced as a refinement of proper coloring. In this approach, for every vertex $v \in V(G)$ of degree at least 2, the neighbors of v must receive at least two distinct colors [1]. This concept has been extensively studied for various graph classes because of its applications in scheduling, communication networks, and frequency assignment problems. Dynamic coloring also provides a stronger local coloring condition compared to ordinary proper coloring and has motivated several extensions and variations in graph coloring theory.

The rainbow vertex connection number, denoted by $rvc(G)$, of a connected graph is the minimum number of colors required to color its vertices such that every pair of vertices is connected by at least one path whose internal vertices have distinct colors, while the end vertices may share the same color [2]. These concepts have attracted considerable attention due to their applications in secure communication and reliable information transfer in networks. Several authors have investigated rainbow vertex connection properties for fan graphs and related graph families in [8, 3]. The study of rainbow vertex connection parameters has expanded rapidly because of their strong relationship with connectivity and path diversity in graphs.

Recently, the concept of rainbow dynamic coloring has emerged by combining the ideas of dynamic coloring and rainbow vertex coloring. This parameter requires a coloring that simultaneously satisfies both the dynamic coloring condition and the rainbow vertex condition. Initial studies on rainbow dynamic coloring for product graphs and corona product graphs were carried out in [7, 9, 10]. In addition, studies on r-dynamic coloring of fan graph families were carried out in [5]. These investigations demonstrate that the structural properties of graphs significantly influence the rainbow dynamic coloring number and motivate further study on specialized graph families. Motivated by these developments, in this paper we investigate rainbow dynamic coloring in fan graphs and star-fan graphs. We determine exact values and establish bounds for the rainbow dynamic coloring number of these graph families. Furthermore, we analyze how the structural properties of fan graphs and star-fan graphs influence the coloring behavior and the existence of rainbow dynamic paths. The obtained results contribute to the study of rainbow dynamic coloring and provide a foundation for further investigations on related graph classes, graph operations, and structural properties associated with rainbow dynamic coloring.

Definition 1. *A fan graph $F_{m,n}$ is defined as the graph join $\overline{K_m} + P_n$, where $\overline{K_m}$ denotes the empty graph on m vertices, P_n denotes the path on n vertices and $+$ denotes the graph join operation. The case $m = 1$ corresponds to the usual fan graph, while $m = 2$ corresponds to the double fan, and so on.*

Throughout this paper, the notation F_n refers to the usual fan graph $F_{1,n}$.

Definition 2. *Let m and n be integers with $m, n \geq 3$. Let S_m be a star graph with $m + 1$ vertices and let F_n be a fan graph with $n + 1$ vertices, $v \in V(F_n)$ and*

v is a vertex of degree n. A star-fan graph is a graph obtained by embedding a copy of F_n into each pendant of S_m, denoted by $S(m, F_n, v_{i,1})$, $i \in [1, m]$, such that the vertex set and the edge set, respectively, are as follows.

- $V(S(m, F_n, v_{i,1})) = \{v_{i,j} \mid i \in [1, m], j \in [1, n+1]\} \cup \{v_{m+1}\}$,
- $E(S(m, F_n, v_{i,1})) = \{v_{m+1}v_{i,1} \mid i \in [1, m]\} \cup \{v_{i,1}v_{i,j} \mid i \in [1, m], j \in [2, n+1]\} \cup \{v_{i,j}v_{i,j+1} \mid i \in [1, m], j \in [2, n]\}$.

Definition 3. *Let m and n be integers with $m, n \geq 3$. Let S_m be a star graph with $m+1$ vertices and let F_n be a fan graph with $n+1$ vertices. Let $v \in V(F_n)$ be a vertex of degree n. A star-fan graph is the graph obtained by embedding a copy of F_n at each pendant vertex of S_m. It is denoted by $S(m, F_n, v_{i,j})$, $i \in [1, m]$, $j \in [2, n]$. The vertex set and edge set of $S(m, F_n, v_{i,j})$ are defined respectively as follows:*

- $V(S(m, F_n, v_{i,j})) = \{v_{i,j} \mid i \in [1, m],\ j \in [1, n+1]\} \cup \{v_{m+1}\}$,
- $E(S(m, F_n, v_{i,j})) = \{v_{m+1}v_{i,j} \mid i \in [1, m]\} \cup \{v_{i,1}v_{i,j} \mid i \in [1, m],\ j \in [2, n+1]\} \cup \{v_{i,j}v_{i,j+1} \mid i \in [1, m],\ j \in [2, n]\}$.

The graphs $S(4, F_4, v_{i,1})$ and $S(4, F_4, v_{i,j})$ are illustrated below.

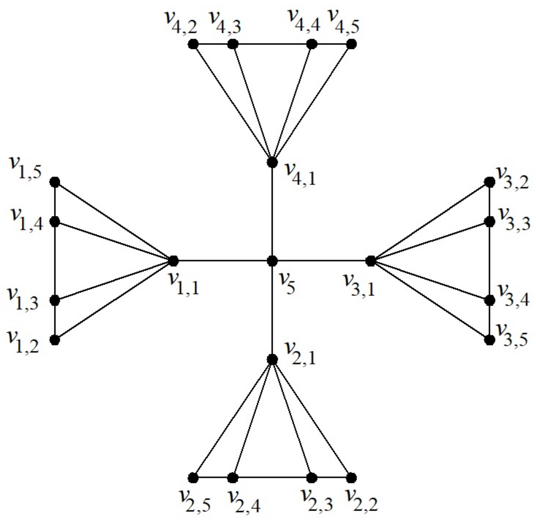

Figure 1: Illustration of the star-fan graph $S(4, F_4, v_{i,1})$.

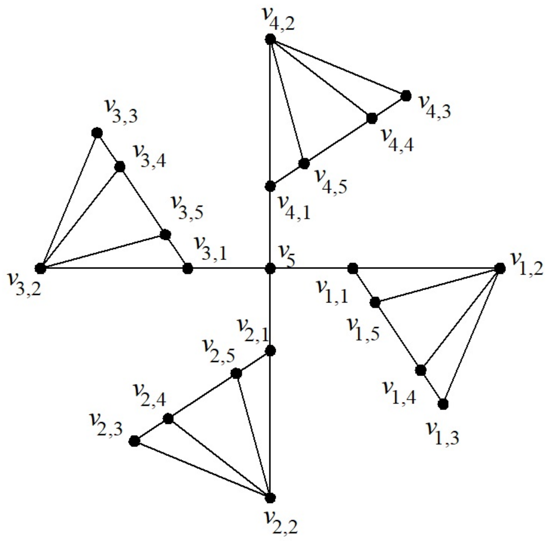

Figure 2: Illustration of the star-fan graph $S(4, F_4, v_{i,j})$.

2. Results

Theorem 1. *For $n \geq 2$, $rdyc(F_{1,n}) = 3$.*

Proof. Let $V(F_{1,n}) = \{c, v_1, v_2, \ldots, v_n\}$, where $v_1, \ldots, v_n$ are the vertices of the path P_n, and c is the central vertex of $\overline{K}_1$. The edge set is $E(F_{1,n}) = \{cv_i \mid 1 \leq i \leq n\} \cup \{v_i v_{i+1} \mid 1 \leq i \leq n-1\}$. At least three colors are needed, one color for the central vertex c and at least two additional colors to properly color the path vertices while ensuring the rainbow dynamic coloring condition along the paths. Hence, we have

$$rdyc(F_{1,n}) \geq 3. \tag{1}$$

We can construct a coloring using exactly three colors as follows, assign $c = 1$, and alternate colors 2 and 3 along the path vertices $c(v_1) = 2,\ c(v_2) = 3,\ c(v_3) = 2, \ldots$ This coloring is dynamic and each pair of vertices is connected by at least one path whose internal vertices receive distinct colors. Therefore,

$$rdyc(F_{1,n}) \leq 3. \tag{2}$$

From (1) and (2),

$$rdyc(F_{1,n}) = 3.$$

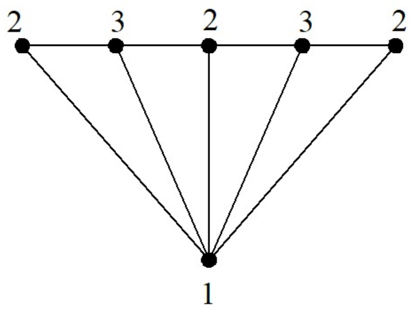

Figure 3: Illustration of $rdyc(F_{1,5}) = 3$.

Theorem 2. *For $n \geq 2$, $rdyc(F_{2,n}) = 3$.*

Proof. Let $V(F_{2,n}) = \{c_1, c_2, v_1, v_2, \ldots, v_n\}$, where c_1 and c_2 are the vertices of $\overline{K}_2$ and $v_1, \ldots, v_n$ are the vertices of the path P_n. The edge set is $E(F_{2,n}) = \{c_i v_j \mid i = 1, 2;\ 1 \leq j \leq n\} \cup \{v_j v_{j+1} \mid 1 \leq j \leq n-1\}$. The central vertices c_1 and c_2 can be assigned the same color since they are not connected. Vertices v_j on every path that is next to c_1 and c_2 cannot use their colors. In order to appropriately color adjacent path vertices, path P_n needs at least two more colors. Therefore, a minimum of $1 + 2 = 3$ colors are required.

$$rdyc(F_{2,n}) \geq 3. \tag{3}$$

We can construct a coloring using exactly three colors as follows, assign $c(c_1) = 1$, $c(c_2) = 1$ and alternate colors 2 and 3 along the path vertices $c(v_1) = 2,\ c(v_2) = 3,\ c(v_3) = 2, \ldots$ This coloring is dynamic and each pair of vertices is connected by at least one path whose internal vertices receive distinct colors. Therefore,

$$rdyc(F_{2,n}) \leq 3. \tag{4}$$

From (3) and (4),

$$rdyc(F_{2,n}) = 3.$$

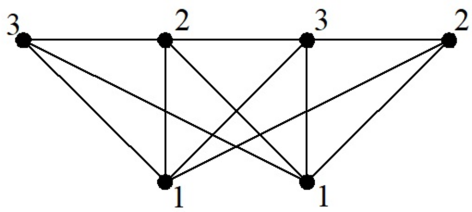

Figure 4: Illustration of $rdyc(F_{2,4}) = 3$.

Theorem 3. *For $m, n \geq 2$, $rdyc(F_{m,n}) = 3$.*

Proof. Let $V(F_{m,n}) = \{v_0, v_1, \ldots, v_{m-1}\}$ be the vertices of $\overline{K}_m$ and $V(P_n) = \{u_1, u_2, \ldots, u_n\}$ are the vertices of the path P_n. At least three colors are needed, assign color 1 to the central vertices $\{v_0, v_1, \ldots, v_{m-1}\}$ and at least two additional colors are required to color the path P_n properly while ensuring the rainbow dynamic condition along the paths. Hence, we have

$$rdyc(F_{m,n}) \geq 3. \tag{5}$$

We can construct a coloring using exactly three colors as follows: $\{v_0, v_1, \ldots, v_{m-1}\}$ as 1, color the vertices of P_n alternately with 2 and 3. This coloring ensures that each vertex of degree at least 2 sees at least two distinct colors in its neighborhood and that every pair of vertices in $F_{m,n}$ is connected by a path whose internal vertices have distinct colors.

$$rdyc(F_{m,n}) \leq 3. \tag{6}$$

From (5) and (6),

$$rdyc(F_{m,n}) = 3$$

Here, we present the results on rainbow dynamic coloring in star-fan graphs.

Theorem 4. *For $m, n \geq 3$, $rdyc(S(m, F_n, v_{i,1})) = m + 1$.*

Proof. Let the vertex set and the edge set be defined as,

- $V(S(m, F_n, v_{i,1})) = \{v_{i,j} \mid i \in [1, m], j \in [1, n+1]\} \cup \{v_{m+1}\}$,
- $E(S(m, F_n, v_{i,1})) = \{v_{m+1}v_{i,1} \mid i \in [1, m]\} \cup \{v_{i,1}v_{i,j} \mid i \in [1, m], j \in [2, n+1]\} \cup \{v_{i,j}v_{i,j+1} \mid i \in [1, m], j \in [2, n]\}$.

Color the vertices of $V(S(m, F_n, v_{i,1}))$ in a rainbow dynamic pattern. For $\{v_{i,j} \mid i \in [1, m], j = 1\}$ as $1, 2, \ldots .m$ for $\{v_{i,j} \mid i \in [1, m], j \in [2, n]\}$ as $i + j - 1$ under the permutation modulo $m + 1$ and v_{m+1} as 1. Based on the color arrangement above,

$$rdyc(S(m, F_n, v_{i,1})) \leq m + 1 \tag{7}$$

To show $rdyc(S(m, F_n, v_{i,1})) \geq m + 1$, suppose that $rdyc(S(m, F_n, v_{i,1})) = m$. Then only m colors are available for a rainbow dynamic coloring of $S(m, F_n, v_{i,1})$. Since the vertices $\{v_{i,1} \mid i \in [1, m]\}$ are adjacent to the central vertex v_{m+1} and must satisfy the dynamic coloring condition, they require distinct colors. Hence all m colors are already used on these vertices. Consequently, the vertex v_{m+1} must receive one of these m colors, which violates the rainbow dynamic coloring condition. Therefore, a rainbow dynamic coloring with only m colors is impossible. Hence,

$$rdyc(S(m, F_n, v_{i,1})) \geq m + 1 \tag{8}$$

From equations (7) and (8), it is clear that

$$rdyc(S(m, F_n, v_{i,1})) = m + 1.$$

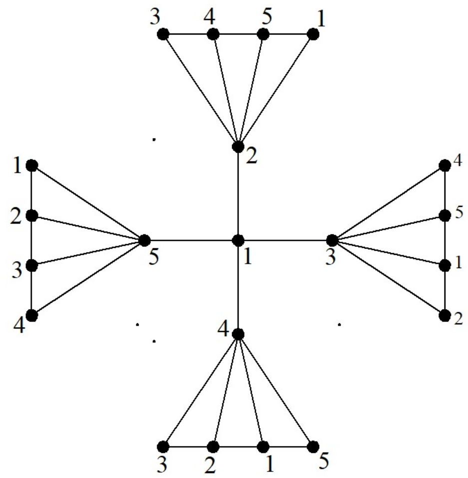

Figure 5: Illustration of $rdyc(S(4, F_4, v_{i,1})) = 5$.

Theorem 5. *For $m, n \geq 3$, $rdyc(S(m, F_n, v_{i,j})) = m + 2$.*

Proof. Let $V(S(m, F_n, v_{i,j})) = \{v_{i,j} \mid i \in [1, m], j \in [1, n+1]\} \cup \{v_{m+1}\}$, $E(S(m, F_n, v_{i,j})) = \{v_{m+1}v_{i,j} \mid i \in [1, m]\} \cup \{v_{i,1}v_{i,j} \mid i \in [1, m], j \in [2, n+1]\} \cup \{v_{i,j}v_{i,j+1} \mid i \in [1, m], j \in [2, n]\}$.

Color the vertices of $V(S(m, F_n, v_{i,j}))$ in a rainbow dynamic pattern. For $\{v_{i,j} \mid i \in [1, m], j = 1\}$ as $i + j$ for $\{v_{i,j} \mid i \in [1, m], j \in [2, n]\}$ as $i + j$ under the permutation modulo $m + 2$ and v_{m+1} as 1. Based on the color arrangement above,

$$rdyc(S(m, F_n, v_{i,j})) \leq m + 2. \tag{9}$$

To show $rdyc(S(m, F_n, v_{i,j})) \geq m + 2$, suppose that $rdyc(S(m, F_n, v_{i,j})) = m + 1$. Then only $m + 1$ colors are available for a rainbow dynamic coloring of $S(m, F_n, v_{i,j})$. The vertices $\{v_{i,j} \mid i \in [1, m]\}$ together with their adjacent vertices must satisfy both the dynamic coloring and rainbow vertex coloring conditions. Consequently, the central vertex v_{m+1} must receive one of the already used colors, which violates the rainbow dynamic coloring condition. Therefore, a rainbow dynamic coloring with only $m+1$ colors is impossible. Hence,

$$rdyc(S(m, F_n, v_{i,j})) \geq m+2. \tag{10}$$

From (9) and (10),

$$rdyc(S(m, F_n, v_{i,j})) = m+2.$$

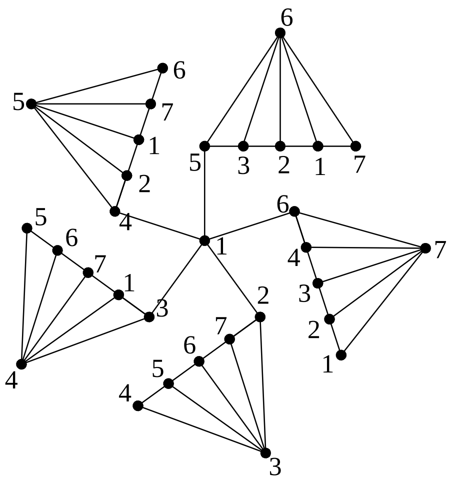

Figure 6: Illustration of $rdyc(S(5, F_5, v_{i,6})) = 7$.

3. Conclusion

In this paper, we have studied rainbow dynamic coloring on certain classes of graphs, namely fan graphs and star-fan graphs. The obtained results illustrate the influence of graph structure on the rainbow dynamic coloring number and contribute to the study of rainbow dynamic coloring parameters. These findings extend earlier investigations on rainbow connection and dynamic coloring in fan graph families [8, 3, 5].

Several open problems remain for future research. One possible direction is to investigate rainbow dynamic coloring for generalized fan graphs and other graph operations where structural interactions may significantly affect the coloring behavior. Another interesting direction is the study of rainbow dynamic coloring for additional graph families and its relationship with existing graph parameters.

Rainbow dynamic coloring also has potential applications in communication networks, particularly in routing, resource allocation, and reliable data transmission systems where diversity of connections and secure communication paths are important [7, 9, 10].

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